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Quantum Theory: The No Cloning Theorem

Tashfeen's master's exit exam presentation included the theoretical proof of the no cloning theorem in quantum physics. It proves the impossibility of cloning arbitrary quantum states. One of the chairs, Dr. Cheng, asked a question which is to be answered here with explicit definitions. Note that we use notation and linearity axiom established in the original presentation. We remind the reader of the key notation,

Definition 1. The *Dirac* or the *bra-ket* notation with the tensor product.

$$|\varphi\rangle\,|\psi\rangle = |\varphi\psi\rangle = \begin{pmatrix}\alpha_\varphi\\ \beta_\varphi\end{pmatrix}\begin{pmatrix}\alpha_\psi\\ \beta_\psi\end{pmatrix} = \begin{pmatrix}\alpha_\varphi\begin{pmatrix}\alpha_\psi\\ \beta_\psi\end{pmatrix}\\ \beta_\varphi\begin{pmatrix}\alpha_\psi\\ \beta_\psi\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\alpha_\varphi\alpha_\psi\\ \alpha_\varphi\beta_\psi\\ \beta_\varphi\alpha_\psi\\ \beta_\varphi\beta_\psi\end{pmatrix}$$

Definition 2. Classical states $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ then,

$$|0\rangle\,|1\rangle = |01\rangle = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\ 0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, |00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Definition 3. All U mapping one quantum state to another have to be linear, i. e., U is a matrix.

$$U(|\varphi\rangle + |\psi\rangle) = U\,|\varphi\rangle + U\,|\psi\rangle$$

Proof by contradiction. There does not exist a universal copier U for any arbitrary quantum state.

Assume for the sake of contradiction that there exists a universal copier U such that $U\,|\psi\rangle\,|0\rangle = |\psi\rangle\,|\psi\rangle$ for all $|\psi\rangle$. Here, U copied the quantum state $|\psi\rangle$ onto $|0\rangle$.

$$\begin{aligned} U\,|\psi\rangle\,|0\rangle &= |\psi\rangle\,|\psi\rangle \\ &= (\alpha\,|0\rangle + \beta\,|1\rangle)(\alpha\,|0\rangle + \beta\,|1\rangle) \\ &= \alpha^2\,|0\rangle\,|0\rangle + \alpha\beta\,|0\rangle\,|1\rangle + \beta\alpha\,|1\rangle\,|0\rangle + \beta^2\,|1\rangle\,|1\rangle \\ &= \alpha^2\,|00\rangle + \alpha\beta\,|01\rangle + \beta\alpha\,|10\rangle + \beta^2\,|11\rangle \end{aligned}$$

But since $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and by the linearity of U,

$$\begin{aligned} U\,|\psi\rangle\,|0\rangle &= U(\alpha\,|0\rangle + \beta\,|1\rangle)\,|0\rangle \\ &= \alpha U\,|0\rangle\,|0\rangle + \beta U\,|1\rangle\,|0\rangle \\ &= \alpha\,|0\rangle\,|0\rangle + \beta\,|1\rangle\,|1\rangle && \text{Since } U\,|1\rangle\,|0\rangle = |1\rangle\,|1\rangle \end{aligned}$$

By transitivity $\alpha^2\,|00\rangle + \alpha\beta\,|01\rangle + \beta\alpha\,|10\rangle + \beta^2\,|11\rangle = \alpha\,|00\rangle + \beta\,|11\rangle$ which is a contradiction. E. g., the reader may try $\alpha = \beta = 1/\sqrt{2}$.

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